12.4

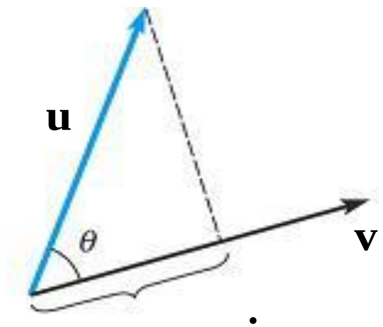
Cross Product

Review:

The **dot product** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

$|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$ $\quad \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$ $\quad$ or $\quad \cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$

$\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$



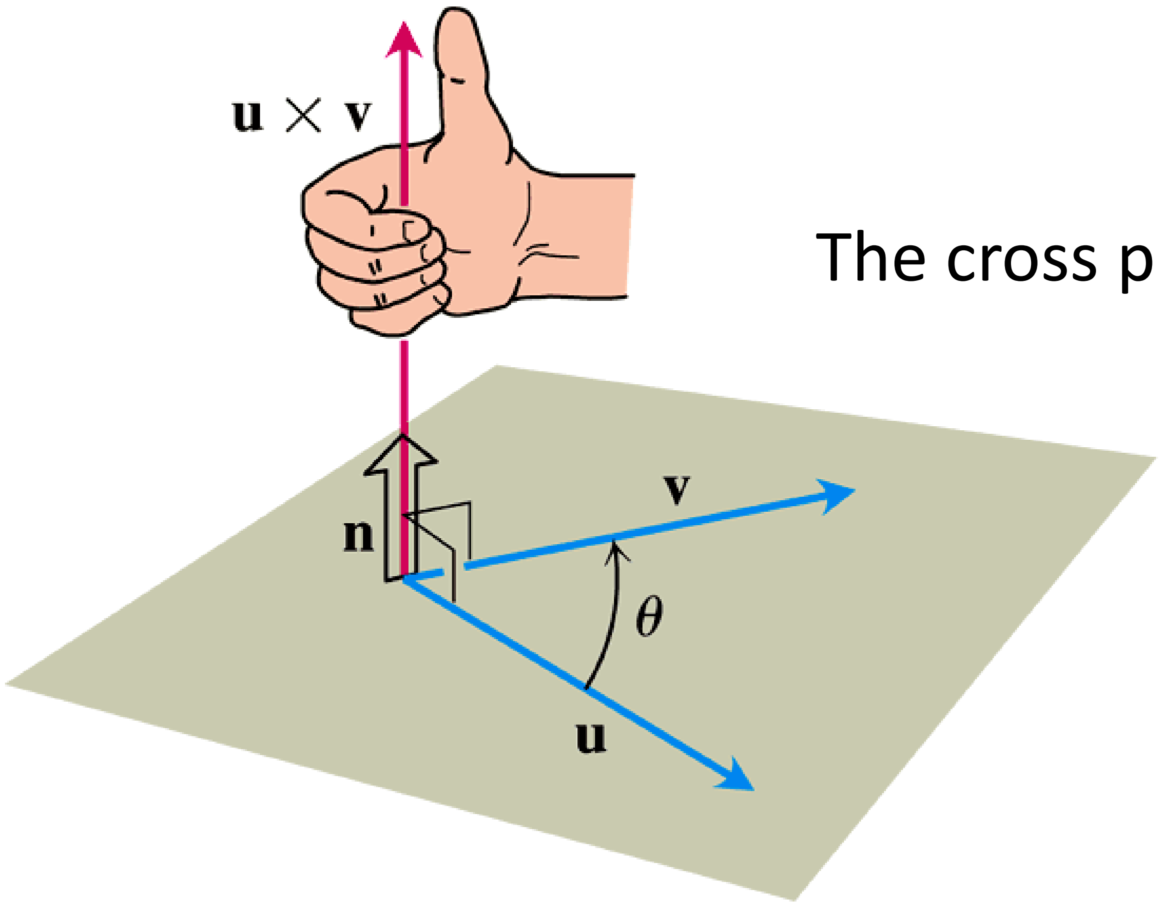
$\text{proj}_{\mathbf{v}}\mathbf{u}$

$$\text{comp}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \qquad \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}$$

cross product $\quad \mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$

$\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. $\quad |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$

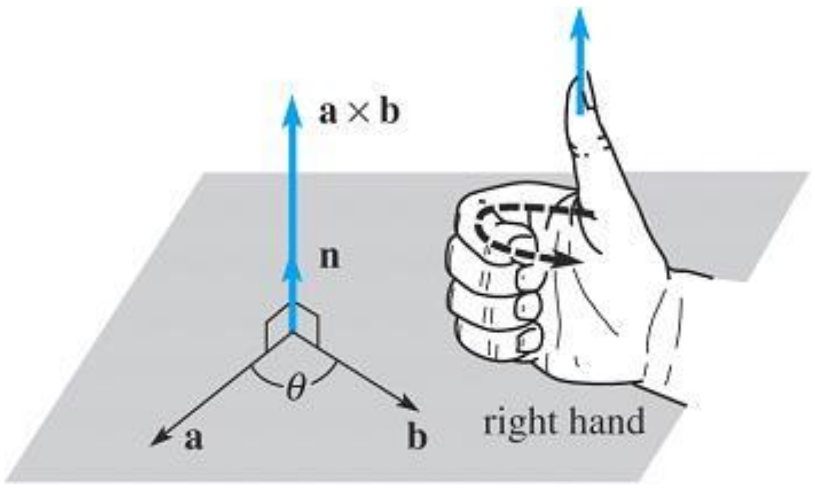
Geometric description of the cross product of the vectors u and v

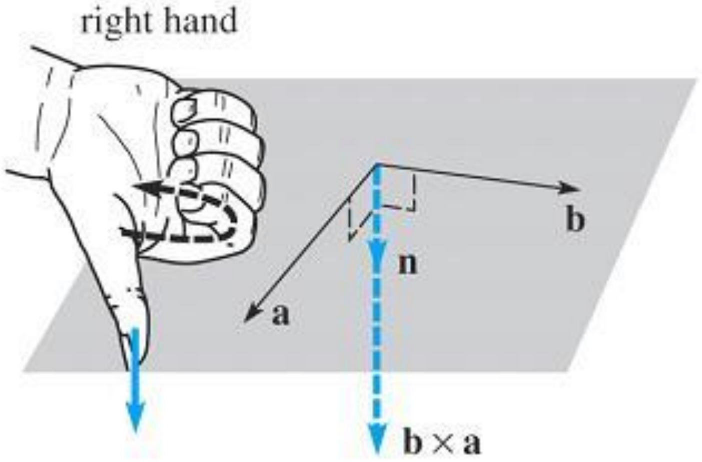


The cross product of two vectors is a vector!

- **u x v** is perpendicular to **u** and **v**
- The length of **u x v** is $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$
- The direction is given by the right hand side rule

Right – hand rule





Place your 4 fingers in the direction of the first vector,

curl them in the direction of the second vector,

Your thumb will point in the direction of the cross product

Algebraic description of the cross product of the vectors u and v

The **cross product** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$$

check $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$ and $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle \cdot \langle u_1, u_2, u_3 \rangle$$

$$= u_2 v_3 u_1 - u_3 v_2 u_1 + u_3 v_1 u_2 - u_1 v_3 u_2 + u_1 v_2 u_3 - u_2 v_1 u_3 = 0$$

similary: $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$

length $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$ is a little messier :

$$|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2 |\mathbf{v}|^2 \sin^2\theta = |\mathbf{u}|^2 |\mathbf{v}|^2 \left(1 - \cos^2\theta\right) = |\mathbf{u}|^2 |\mathbf{v}|^2 \left(1 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{|\mathbf{u}|^2 |\mathbf{v}|^2}\right) = |\mathbf{u}|^2 |\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

now need to show that $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2 |\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ (try it..)

An easier way to remember the formula for the cross products is in terms of *determinants:*

2x2 determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$

3x3 determinants: An example

Copy 1st 2 columns

$$\begin{vmatrix} 1 & 6 & -2 \\ 3 & -1 & 3 \\ 4 & 5 & 2 \end{vmatrix} \qquad \begin{vmatrix} 1 & 6 & -2 \\ 3 & -1 & 3 \\ 4 & 5 & 2 \end{vmatrix} \begin{matrix} 1 & 6 \\ 3 & -1 \\ 4 & 5 \end{matrix} \qquad \begin{pmatrix} \text{sum of} \\ \text{forward} \\ \text{diagonal} \\ \text{products} \end{pmatrix} - \begin{pmatrix} \text{sum of} \\ \text{backward} \\ \text{diagonal} \\ \text{products} \end{pmatrix}$$

$$\text{determinant} = (-2 + 72 - 30) - (8 + 15 + 36) = 40 - 59 = -19$$

recall: $\mathbf{u}\times\mathbf{v}=\left\langle u_2v_3-u_3v_2, u_3v_1-u_1v_3, u_1v_2-u_2v_1\right\rangle$

now we claim that $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}\begin{matrix}\mathbf{i} & \mathbf{j}\\ u_1 & u_2\\ v_1 & v_2\end{matrix}$$

$$=\mathbf{i}u_2v_3+\mathbf{j}u_3v_1+\mathbf{k}u_1v_2-\mathbf{k}u_2v_1-\mathbf{i}u_3v_2-\mathbf{j}u_1v_3$$

$$\mathbf{u}\times\mathbf{v}=(u_2v_3-u_3v_2)\mathbf{i}-(u_1v_3-u_3v_1)\mathbf{j}+(u_1v_2-u_2v_1)\mathbf{k}$$

$$\mathbf{u}\times\mathbf{v}=\left\langle u_2v_3-u_3v_2, u_3v_1-u_1v_3, u_1v_2-u_2v_1\right\rangle$$

Example: Let $\mathbf{u} = \langle 1, -2, 1 \rangle$ and $\mathbf{v} = \langle 3, 1, -2 \rangle$ Find $\mathbf{u} \times \mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix} =$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix} \begin{matrix} \mathbf{i} & \mathbf{j} \\ 1 & -2 \\ 3 & 1 \end{matrix}$$

$$\mathbf{u} \times \mathbf{v} = (4 - 1)\mathbf{i} \ + (3 + 2)\mathbf{j} \ + (1 + 6)\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \langle 3, 5, 7 \rangle$$

Geometric Properties of the cross product:

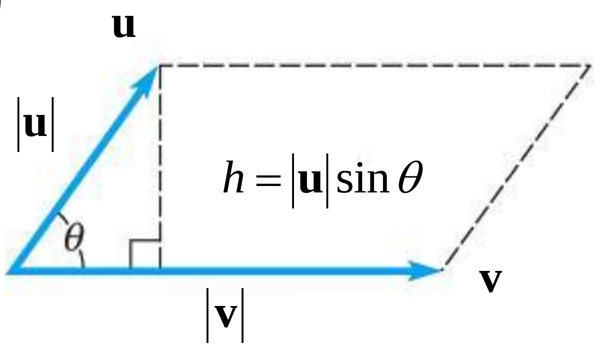
Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

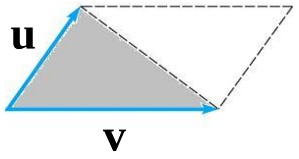
2. $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$

3. $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other (they are parallel)

4. $|\mathbf{u}\times\mathbf{v}| =$ area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.



5. $\frac{1}{2}|\mathbf{u}\times\mathbf{v}| =$ area of the triangle having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.



Problem: Compute the area of the triangle with vertices (2,3,4), (1,3,2), (3,0,-6)

Two sides are: $\mathbf{u} = \langle 1, 0, 2 \rangle$ and $\mathbf{v} = \langle -1, 3, 10 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ -1 & 3 & 10 \end{vmatrix}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ -1 & 3 & -2 \end{vmatrix} \begin{matrix} \mathbf{i} & \mathbf{j} \\ 1 & 0 \\ -1 & 3 \end{matrix}$$

$$= (0-6)\mathbf{i} + (-2-10)\mathbf{j} + (3-0)\mathbf{k}$$

$$= -6\mathbf{i} - 12\mathbf{j} + 3\mathbf{k}$$

$$= \langle -6, -12, 3 \rangle$$

$$|\mathbf{u} \times \mathbf{v}| = \sqrt{36 + 144 + 9} = \sqrt{189}$$

$$\text{area} = \frac{3}{2}\sqrt{21}$$

Algebraic Properties of the cross product:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left\langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \right\rangle$$

Let $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ be vectors and let c be a scalar.

1. $\mathbf{u}\times\mathbf{v} = -\left(\mathbf{v}\times\mathbf{u}\right)$

2. $\mathbf{u}\times\left(\mathbf{v}+\mathbf{w}\right) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$

3. $\left(c\mathbf{u}\right)\times\mathbf{v} = \mathbf{u}\times\left(c\mathbf{v}\right) = c\left(\mathbf{u}\times\mathbf{v}\right)$

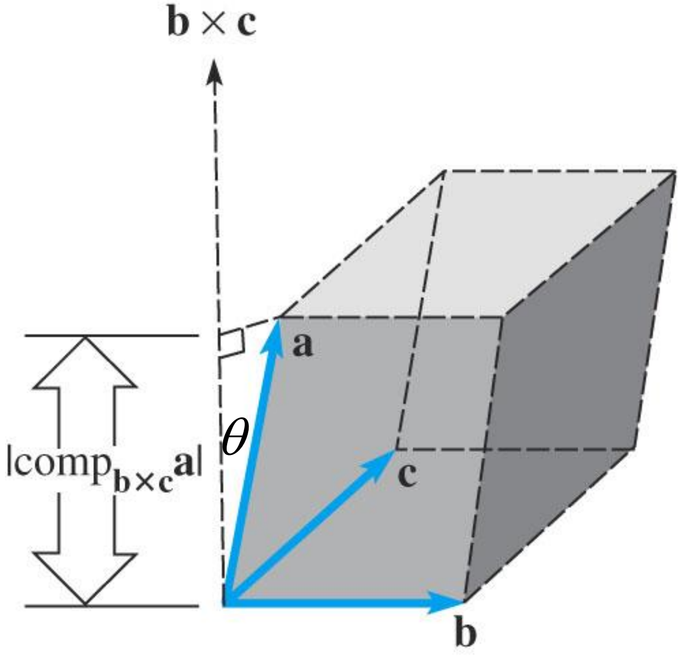
4. $\mathbf{0}\times\mathbf{v} = \mathbf{v}\times\mathbf{0} = \mathbf{0}$

5. $(\mathrm{c}\mathbf{v})\times\mathbf{v} = \mathbf{0}$

6. $\mathbf{u}\cdot\left(\mathbf{v}\times\mathbf{w}\right) = \left(\mathbf{u}\times\mathbf{v}\right)\cdot\mathbf{w}$

7. $\mathbf{u}\times\left(\mathbf{v}\times\mathbf{w}\right) = \left(\mathbf{u}\cdot\mathbf{w}\right)\mathbf{v}-\left(\mathbf{u}\cdot\mathbf{v}\right)\mathbf{w}$

Volume of the parallelepiped determined by the vectors $\mathbf{a}, \mathbf{b},$ and $\mathbf{c}$.



$$\text{Area of the base} = |\mathbf{b}\times\mathbf{c}|$$

$$\text{Height} = \text{comp}_{\mathbf{b}\times\mathbf{c}}\mathbf{a} = |\mathbf{a}|\cos\theta$$

$$\text{Volume} = |\mathbf{b}\times\mathbf{c}||\mathbf{a}|\cos\theta$$

$$\text{Volume} = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$ ← this stands for absolute value

$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is called the **scalar triple product**

The vectors are in the same plane (**coplanar**) if the scalar triple product is 0.

Problem: Compute the volume of the parallelepiped spanned by the 3 vectors

$$\mathbf{u} = \langle 1, 0, 2 \rangle \ , \ \mathbf{v} = \langle -1, 3, -2 \rangle \ \text{ and } \mathbf{w} = \langle -1, 3, -4 \rangle$$

Solution:

From slide 10: $\mathbf{u} \times \mathbf{v} = \langle -6, 0, 3 \rangle$

$$\langle -6, 0, 3 \rangle \cdot \langle -1, 3, -4 \rangle$$

$$= 6 - 12 = -6$$

Volume = 6

Quicker:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathrm{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot \langle w_1, w_2, w_3 \rangle$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{matrix} \mathbf{i} & \mathbf{j} \\ u_1 & u_2 \\ v_1 & v_2 \end{matrix} \cdot \langle w_1, w_2, w_3 \rangle = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

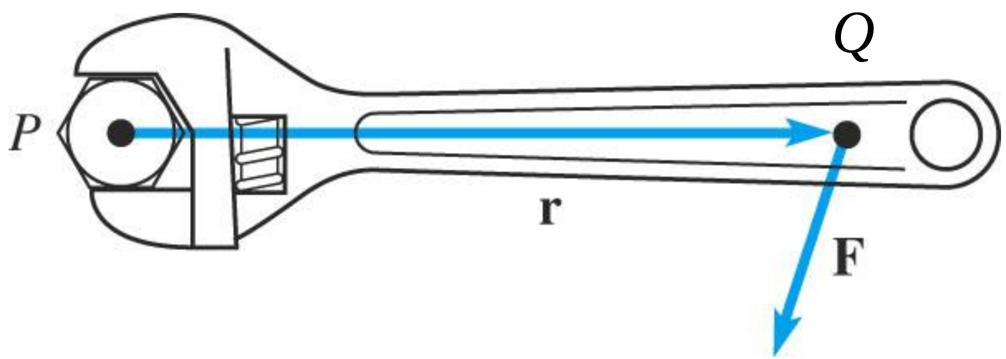
$$(\mathbf{u} \times \mathbf{v}) \cdot \mathrm{w} = \begin{vmatrix} -1 & 3 & -4 \\ 1 & 0 & 2 \\ -1 & 3 & -2 \end{vmatrix} = (0 - 6 - 12) - (-6 - 6 - 0)$$

$$= -6$$

Triple scalar product

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathrm{w} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

In physics, the cross product is used to measure **torque**.



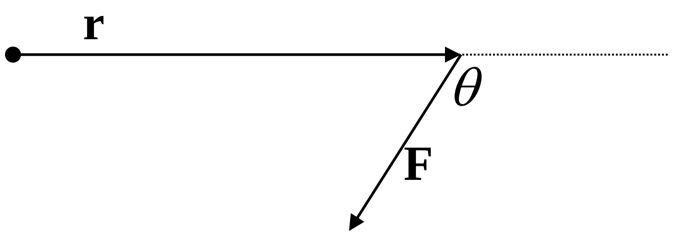
Consider a force **F** acting on a rigid body at a point given by a position vector **r**.

The torque (τ) measures the tendency of the body to rotate about the origin (point P)

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}|\sin\theta$$

(θ is the angle between the force and position vectors)



12.5

Lines and Planes

Recall how to describe lines in the plane (e.g. tangent lines to a graph):

$y = mx + b$ m is the slope b is the y intercept

Point slope formula: $$\frac{y - y_0}{x - x_0} = m$$ (x_0, y_0) is on the line

Two point formula: $$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$ (x_0, y_0) and (x_1, y_1) are on the line

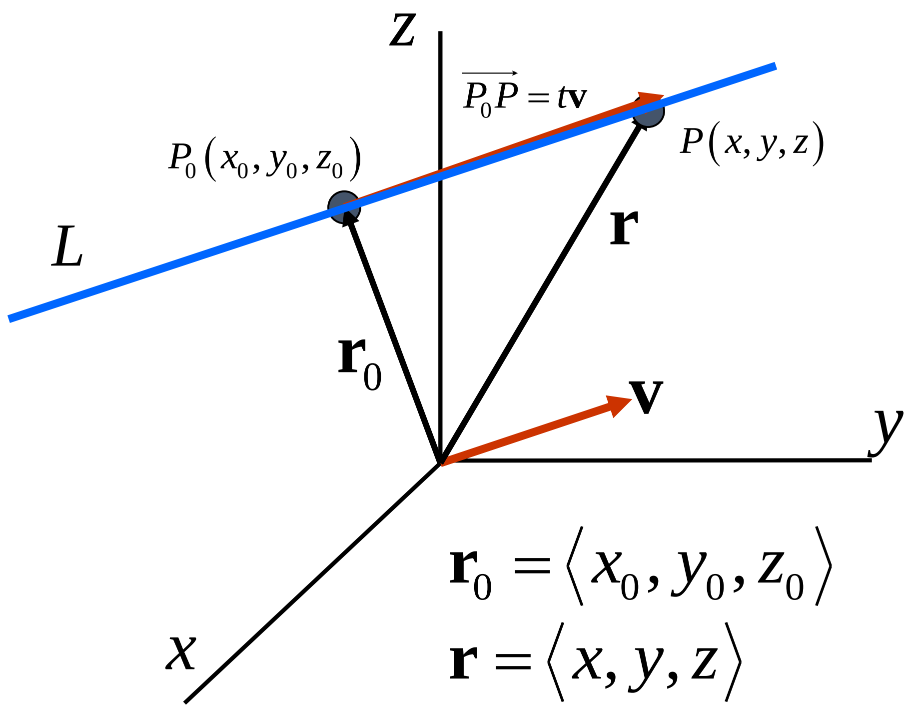
Equations of Lines and Planes

In order to find the equation of a line, we need :

A) a point on the line $P_0\left(x_0, y_0, z_0\right)$

B) a direction vector for the line $\mathbf{v} = \langle a, b, c \rangle$

$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ <u>**vector equation**</u> of line L



$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$$

$$\mathbf{r} = \langle x, y, z \rangle$$

$$\overrightarrow{P_0P} = t\mathbf{v} = t\langle a, b, c \rangle \text{ for some } t$$

Here $\mathbf{r}_0$ is the vector from the origin to a *specific* point P_0 on the line

$\mathbf{r}$ is the vector from the origin to a *general* point $P = (x, y, z)$ on the line

$\mathbf{v}$ is a vector which is *parallel* to a vector that lies on the line

$\mathbf{v}$ is *not* unique: $2\mathbf{v}$, or $-\mathbf{v}$ will also do

vector equation of the line *L*

$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ or $\boxed{\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle}$

equating components we get the **parametric equations** of the line *L*

$$\boxed{x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct}$$

Solving for t we get the **symmetric equations** of the line *L*

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Problem:

Find the parametric equations of the line containing $\mathrm{P}_0 = (5,1,3)$ and $\mathrm{P}_1 = (3,-2,4)$.

A) a point on the line $P_0(x_0, y_0, z_0)$ choose $\mathrm{P}_0 = (5,1,3)$ (could also choose $\mathrm{P}_0 = (3,-2,4)$)

B) a direction vector for the line $\mathbf{v} = \langle a,b,c \rangle$

$$\mathbf{v} = \overrightarrow{P_0P_1} = P_1 - P_0 = \langle 3-5, -2-1, 4-3 \rangle = \langle -2,-3,1 \rangle$$

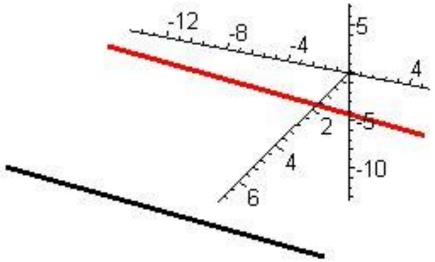
or $\mathbf{v} = \langle 2,3,-1 \rangle$

$$\langle x,y,z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a,b,c \rangle = \langle 5,1,3 \rangle + t\langle 2,3,-1 \rangle$$

The line is: $x = 5+2t, \quad y = 1+3t, \quad z = 3-t$

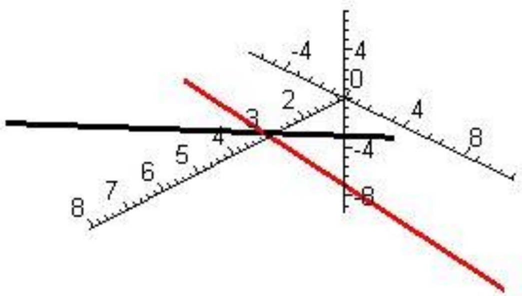
Two lines in 3 space can interact in 3 ways:

A) **<u>Parallel Lines</u>** -



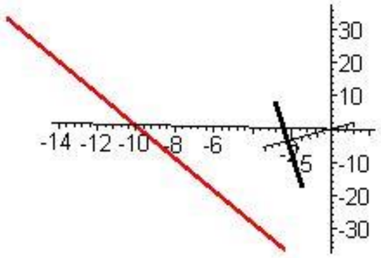
their direction vectors are scalar multiples of each other

B) **<u>Intersecting Lines</u>** -



there is a specific *t* and *s*, so that the lines share the same point.

C) **<u>Skew Lines</u>** -



their direction vectors are **not** parallel and there is **no** values of *t* and *s* that make the lines share the same point.

Problem: Determine whether the lines L_1 and L_2 are parallel, skew or intersecting. If they intersect, find the point of intersection.

$$L_1 \ : \ x = 3 - t, \ \ y = 5 + 3t, \ \ z = -1 - 4t \qquad L_2 \ : \ x = 8 + 2s, \ \ y = -6 - 4s, \ \ z = 5 + s$$

Set the x coordinate equal to each other: $3 - t = 8 + 2s$, or $2s + t = -5$

Set the y coordinate equal to each other: $5 + 3t = -6 - 4s$, or $4s + 3t = -11$

We get a system of equations:

$$\begin{aligned} 2s + t &= -5 \\ 4s + 3t &= -11 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 4s + 2t &= -10 \\ 4s + 3t &= -11 \end{aligned} \qquad \begin{aligned} t &= -1 \\ s &= -2 \end{aligned}$$

Check to make sure that the z values are equal for this t and s.

$$-1 - 4t = 5 + s$$

$$-1 - 4(-1) = 5 + (-2)$$

$$3 = 3 \ \ \text{check}$$

Find the point of intersection using L_1 :

$$\begin{aligned} x &= 3 - (-1) \\ y &= 5 + 3(-1) \\ z &= -1 - 4(-1) \end{aligned} \qquad (4, 2, 3)$$