Numerical evaluation of the localizable entanglement in 1D spin chain model

Dongkeun Lee[1 2 *] Wonmin Son[1 †]

[1] *Department of Physics, Sogang University, 35, Baekbeom-ro, Mapo-gu, Seoul 04107, Republic of Korea*
[2] *Research Institute for Basic Science, Sogang University, 35, Baekbeom-ro, Mapo-gu, Seoul 04107, Republic of Korea*

Abstract. The Localizable entanglement(LE), suggested by Verstraete(2004), is a measure of multipartite entanglement which is into pairwise entanglement between two spins. The property of the LE is that the tightly lower bound of the LE measured by the concurrence is the maximal correlation function between two parties, which implies that LE can detect the many-body physics more subtly than the maximal correlation function does. Generally speaking, it is non-trivial to calculate LE numerically due to the difficulty for the optimal choices of measurement bases. Here, a new approach is proposed to overcome a limited number of parties, which is analogous to the variational matrix product states(VMPS) method.

Keywords: Localizable Entanglement, the variational matrix product states(VMPS) method

1 Definition of the localizable entanglement

Let us consider a pure state $|\psi\rangle$ of N spins. Then, the **localizable entanglement** (LE) is defined as the maximal amount of averaged entanglement that can be created or localizable between the spins i and j by performing local measurements on the other spins. Note that the remaining spins are locally measured rather than traced out. To be more specific, every measurement $\mathcal{M}$ specifies a state ensemble $\mathcal{E}_{\mathcal{M}} := \{p_s, |\psi_s^{ij}\rangle\}$. Here, p_s denotes the probability to obtain the (normalized) two-spin state $|\psi_s^{ij}\rangle$ for the outcome $\{s\}$ of the measurements on the $N-2$ remaining spins.
The average entanglement for a specific $\mathcal{M}$ is

$$L_{i,j}^{\mathcal{M},E}(|\psi\rangle) := \sum_s p_s E(|\psi_s^{ij}\rangle), \tag{1}$$

where $E(|\psi\rangle_s^{ij})$ is the entanglement of $|\psi\rangle_s^{ij}$.
The **localizable entanglement** is defined as the largest possible average entanglement,

$$L_{i,j}^{\mathcal{C},E}(|\psi\rangle) := \sup_{\mathcal{M}\in\mathcal{C}} \sum_s p_s E(|\psi_s^{ij}\rangle), \tag{2}$$

with $\mathcal{C}$ denoting the class of allowed measurements. The measurement $\mathcal{M}$ which maximizes the average entanglement is called the *optimal basis.*[**?**]

2 The physical properties of the LE

The one of the main properties in the LE is that the tightly lower bound of the localizable entanglement, measured by the concurrence, is the maximal correlation function,

$$\max_{\vec{a},\vec{b}} |Q_{AB}^{ij}(|\psi\rangle)| \le L_{i,j}^{C}(|\psi\rangle), \tag{3}$$

$$Q_{AB}^{ij}(|\psi\rangle) \equiv \mathrm{tr}[\rho(S_A^i \otimes S_B^j)] - \mathrm{tr}[\rho(S_A^i \otimes I)]\mathrm{tr}[\rho(I \otimes S_B^j)]. \tag{4}$$

Here, we parametrize the operator $S_A \equiv \vec{a}\cdot\vec{\sigma}$ and $S_B \equiv \vec{b}\cdot\vec{\sigma}$ with the vector $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ and the unit vectors $\vec{a}$ and $\vec{b}$ represent the direction in a 3D real space. It is similar that the entanglement length ξ_E defined just as the correlation length ξ_C from the correlation function is always larger than the correlation length[**?**]. Due to the choice of the optimal basis, the localizable entanglement $L_{i,i+1}^C$ exhibit a discontinuity at the critical point $\Delta = -1$ for 1D XXZ model.

3 The LE measured by the concurrence

By the definition of the localizable entanglement, any measure of bipartite entanglement can be a proper candidate. Among these entanglemt measures, we select the concurrence because its definition becomes concise in the case of the two-qubit pure states, Hence, we confine the conditions of the many-body state to the pure one for the two-level systems. In general, a pure state for two-level systems can be expressed as

$$|\psi\rangle = \sum_{s_1,s_2,\cdots,s_N} c_{s_1 s_2 \cdots s_N} |s_1 s_2 \cdots s_N\rangle, \tag{5}$$

with $s_l \in \{0,1\}$ for all $l \le N$ and the $|s_l\rangle$ are in the σ_z basis, which is now regarded as the computational basis. Alternatively, we can utilize the matrix product states(MPS) language, approximately represents the many-body ground states,

$$|\psi\rangle = \sum_{s_1,s_2,\cdots,s_N} \mathrm{Tr}\left[A_1^{s_1} A_2^{s_2} \cdots A_N^{s_N}\right] |s_1 s_2 \cdots s_N\rangle, \tag{6}$$

where $A_l^{s_l}$ is the $D \times D$ matrices. This MPS acounts for periodic boundary condsitions(PBC).

Moreover, the local measurement must be performed on the remaining spins and we are going to use the projective measurement, also called the von Neumann measurement $\mathcal{M} = \{|+\rangle\langle+|, |-\rangle\langle-|\}$, that are performed on

*dglee90@gmail.com
†sonwm71@gmail.com

the lth spin in the basis,

$$|+\rangle \equiv \sum_{s_l} u^{+}_{s_l}|s_l\rangle = \cos(\theta_l/2)|0\rangle + e^{i\varphi_l}\sin(\theta_l/2)|1\rangle \quad (7)$$

$$|-\rangle \equiv \sum_{s_l} u^{-}_{s_l}|s_l\rangle = -e^{-i\varphi_l}\sin(\theta_l/2)|0\rangle + \cos(\theta_l/2)|1\rangle. \quad (8)$$

After the projective measurements on spins except $i$the and jth spins, the number of measurement outcomes is d^{N-2} and all the possible outcomes are denoted by $\mathbf{m} := \{m_1 m_2 \dots m_{i-1} m_{i+1} \dots m_{j-1} m_{j+1} \dots m_N\}$ with $m_l \in \{+,-\}$. The post-measurement state becomes

$$|\phi^{ij}_{\mathbf{m}}\rangle \equiv \langle \mathbf{m}|\psi\rangle$$
$$= \sum_{s_i,s_j}\sum_{\mathbf{m}} \mathrm{Tr}\left[\bar{A}^{m_1}_1 \cdots \bar{A}^{m_{i-1}}_{i-1} A^{s_i}_i \bar{A}^{m_{i+1}}_{i+1} \cdots A^{s_j}_j \cdots \bar{A}^{m_N}_N\right]|s_i s_j\rangle,$$

where

$$|\mathbf{m}\rangle = |m_1 m_2 \dots m_{i-1} m_{i+1} \dots m_{j-1} m_{j+1} \dots m_N\rangle \quad (9)$$

$$= \sum_{\{s_l\}_{l\neq i,j}} u^{m_1}_{s_1} \cdots u^{m_{i-1}}_{s_{i-1}} u^{m_{i+1}}_{s_{i+1}} \cdots u^{m_{j-1}}_{s_{j-1}} u^{m_{j+1}}_{s_{j+1}} \cdots u^{m_N}_{s_N} \quad (10)$$

$$|s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} s_{j+1} \cdots s_N\rangle,$$

and $$\bar{A}^{m_l} \equiv \sum_{s_l} A^{s_l}\left(u^{m_l}_{s_l}\right)^* |s_l\rangle \quad (11)$$

If the post-measurement state is simply $|\phi^{ij}_{\mathbf{m}}\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$, one can easily calculate the concurrence as $C(|\phi^{ij}_{\mathbf{m}}\rangle) = 2|ad-bc|$. With these notations, the localizable entanglement can be calculated by optimizing the averaged concurrence $\bar{L}^{\mathcal{M},C}_{i,j} = \sum_{\mathbf{m}} p_{\mathbf{m}} C\left(|\phi^{ij}_{\mathbf{m}}\rangle/\sqrt{p_{\mathbf{m}}}\right)$ with respect to the matrices $u^{m_l}_{s_l}$ for all $l \leq N$ but i and j. For instance, when $N=3$ GHZ state $|GHZ\rangle$ is given, $L^{\mathcal{M},C}_{1,2}(|GHZ\rangle) = 1$, whereas $C_{1,2}(|GHZ\rangle) = 0$.

4 The new approach and the result

In general, it is not straightforward to calculate the localizable entanglement. Especially when the number of spins is large. As the spins number increases, it is hard to evaluate the optimization problem in the definition of the localizable entanglement. The exact diagonalization, a numerical method, is available up to near N=20. Moreover, the number of all possible measurement outcomes is exponentially large with respect to the number of spins, which causes the sum of an exponential number of terms d^{N-2}(d: the dimensions of the spin). In Popp's paper[**?**], Monte Carlo method is proposed to overcome the exponential summation, as though statistical error becomes large at the critical point.

We suggest an alternative scheme to resolve the first difficulty. Let us suppose that there is a function f of multi-vectors such that

$$f(\vec{X}_1, \vec{X}_2, \dots, \vec{X}_N), \quad (12)$$

where the elements in the vectors $\vec{X}_i$ are independent variables. The main idea of the scheme for maximizing the function f is that we find the maximum value of the function with respect to the only one variable, otherwise the others are fixed. That is, in the numerical algorithm, arbitrary numbers are inserted in each variable except one which we want to optimize the function with respect to. To be more specific, the random vectors $\vec{X}_{2o}, \cdots, \vec{X}_{No}$ being fixed, we maximize the function of $\vec{X}_1$ and find the maximal point $\vec{X}_{1c}$. Next, the vector $\vec{X}_1$ is upgraded to $\vec{X}_{1c}$ and maximize the function with respect to the variable $\vec{X}_2$ as the vectors $\vec{X}_{3o}, \cdots, \vec{X}_{No}$ are still stationary. We consecutively maximize the function with respect to the one variable $\vec{X}_i$ and upgrade $\vec{X}_{io}$ to $\vec{X}_{ic}$ until the integer i approaches to N. This whole procedure is iterated until the maximum of the function converges.

The calculation of the localizable entanglement for a many-body state is implemented based on the above idea. The function f and the vectors $\vec{X}_l$ correspond to the averaged entanglement and the direction of the measurement basis on the lth spin, respectively. According to the definition of the localizable entanglement, any measure of bipartite entanglement can be a proper candidate. We select the concurrence among these entanglement measures because its definition becomes concise in the case of the two-qubit pure states, which helps to simplify and boost up the calculation of the optimization. Hence, we confine the conditions of the many-body state to the pure one for the two-level systems. Also, we choose the local measurement as the projective one. Then, we simplify the formula of the averaged concurrence in terms of the ith vector such that

$$f(\vec{X}_l) := \sum_{\mathbf{m}=1}^{2^{N-2}} \sqrt{\vec{X}^T_l \cdot \mathcal{L}^{[l]}_{\mathbf{m}} \cdot \vec{X}_l}, \quad (13)$$

where the 4-by-4 symmetric matrix $\mathcal{L}^{[l]}_{\mathbf{m}}$ contains the elements related to $A^{s_i}_i$, $A^{s_j}_j$, $A^{s_l}_l$, and $\bar{A}^{m_k}_k$ for all k except i,j, and l and $\vec{X}^T_l := \frac{1}{\sqrt{2}}(1, \cos\phi_l \sin\theta_l, \sin\phi_l \sin\theta_l, \cos\theta_l)$. The maximization of this function with constraint $\vec{X}^T_l \cdot \vec{X}_l = 1$ and the update of these vectors being iterated by using the built-in function in Mathematica, the maximum will finally converge the real number. To test the accuracy of this algorithm, we simulate the one-dimensional ising model with the transverse field, since the localizable entanglement measured by the concurrence is the same as the two-point correlated function Q_{xx} by Popp's paper. The ground-energy state is obtained through the variational method in matrix product states representation. The Fig.2 shows the localizable entanglement for the ground state of the 1D ising model at the critical point concerning the quantum phase transition and the error of $L^C_{1,2}$ compared to the $Q^{1,2}_{xx}$ is around 10^{-6}.

Still, the summation amount in terms of an exponential number of measurement outcomes has yet to be surmounted. To overcome this impediment, we are going to choose more weighted measurement outcomes strategically. That is, the summand in Eq.(**??**) has diffirenet values for each measurement outcomes when vecotrs $\vec{X}_i$

are fixed and we will strategically select some measurement outcomes $\bar{\mathbf{m}}$ that make the largest eigenvalues of the matrices $\mathcal{L}^{[i]}_{\bar{\mathbf{m}}}$ relatively larger than those of the other matrices. Consequently, we are going to calculate the summation Eq.(**??**) over chosen measurement outcomes $\bar{\mathbf{m}}$ up to a small number $M < d^{N-2}$

$$f(\vec{X}_i) \simeq \sum_{\bar{m}=1}^{M} \sqrt{\vec{X}_i^T \cdot \mathcal{L}^{[i]}_{\bar{\mathbf{m}}} \cdot \vec{X}_i}, \tag{14}$$

which is the main idea that we are going to research further. After maximizing Eq.(**??**) for all $i \leq N$, we will compare its maximal value with that of Eq.(**??**) to check the efficiency of this approximation for small particle number ($N \leq 16$) and then simulate the localizable entanglement in the case of a large number of the particles. Based on this approach, we are going to research other many-body systems, especially the two-dimensional systems. That is because more two-dimensional systems have topological properties than one-dimensional ones. We anticipate that the localizable entanglement would catch exotic phase transitions in 2D systems.

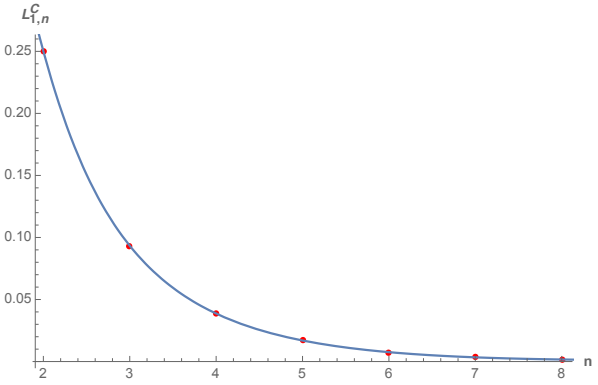

Figure 1: The localizable entanglement $L^C_{i,i+n}$(the red points) measured by the concurrence between first spin and the nth spin: its optimal measurement is in the σ_x-basis for 1D ising model and the LE is the same as the two-point correlated function Q^{ij}_{xx}. At its critical point, $L^C_{i,i+n}$ has a tendency like the power-law decay $n^{-1/4}$(the blue line).

References

[1] F. Verstraete, M. Popp, and J. I. Cirac. Entanglement versus correlations in spin systems *Phys. Rev. Lett* **92**, 027901, 2004

[2] M. Popp, F. Verstraete, and J. I. Cirac. Localizable entanglement *Phys. Rev. A* **71**, 042306, 2005

[3] L. Amico, R. Fazio, A. Osterloh, and V. Vedral. Entanglement in many-body systems *Rev. Mod. Phys.* **80**, 517, 2008

[4] F. Verstraete, M. A. Martín-Delgado, J. I. Cirac. Diverging Entanglement Length in Gapped Quantum Spin Systems *Phys. Rev. Lett* **92**, 087201, 2004

[5] W. K. Wooters. Entanglement of Formation of an Arbitrary State of Two Qubits *Phys. Rev. Lett.* **80**, 2245, 1998