p361 Section 5.6: Differential Equations: Growth and Decay

In this section, you will learn how to solve a more general type of differential equation. You will need to rewrite the equation so that each variable occurs on only one side of the equation.

Example 1: Solving a Differential Equation

$$y' = \frac{2x}{y}$$

Multiply both sides by y to get each variable on a separate side of the equation

$$yy' = 2x$$

Set up to integrate both sides of the equation

$$\int yy'dx = \int 2xdx$$

Replace y'dx with dy

$$\int ydy = \int 2xdx$$

Integrate each side - you don't have to include a C on both sides

$$\frac{1}{2}y^2 = x^2 + C$$

Write the equation as a general solution (equal to C)

$$\frac{1}{2}y^2 - x^2 = C$$

$$y^2 - 2x^2 = C$$

** You could have also written the equation as follows:

$$\frac{dy}{dx} = \frac{2x}{y}$$

$$ydy = 2xdx$$

$$\int ydy = \int 2xdx$$

Solve the differential equation

#3. $\frac{dy}{dx} = y + 2$

$$\frac{dy}{y+2} = dx$$

$$\int \frac{1}{y+2} dy = \int dx$$

$$\ln|y+2| = x + C$$

$$y + 2 = e^{x+C}$$

$$y + 2 = e^x e^C$$

$$y + 2 = Ce^x e$$

$$y + 2 = Ce^x$$

$$y = Ce^x - 2$$

** Since e is a constant, you can "drop" it - it becomes part of C

#7. $y' = \sqrt{xy}$

$$\frac{y'}{y} = x^{1/2}$$

$$\int \frac{1}{y} y' dx = \int x^{1/2} dx$$

$$\int \frac{1}{y} dy = \int x^{1/2} dx$$

$$\ln|y| = \frac{x^{3/2}}{3/2} + C$$

$$\ln|y| = \frac{2}{3} x^{3/2} + C$$

$$y = e^{(2/3)x^{3/2} + C}$$

$$y = e^C e^{(2/3)x^{3/2}}$$

$$y = Ce^{(2/3)x^{3/2}}$$

#9. $(1+x^2)y' - 2xy = 0$

$$y' = \frac{2xy}{(1+x^2)}$$

$$\frac{y'}{y} = \frac{2x}{(1+x^2)}$$

$$\int \frac{y'}{y} dx = \int \frac{2x}{(1+x^2)} dx$$

$$\int \frac{1}{y} dy = \int \frac{2x}{(1+x^2)} dx$$

$$\ln y = \ln(1+x^2) + C$$

$$\ln y = \ln(1+x^2) + \ln C$$

$$\ln y = \ln C(1+x^2)$$

$$y = C(1+x^2)$$

#15. A differential equation, a point and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passe through the indicated point (0, 0) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution.

** The slope field is on the next page

$$\frac{dy}{dx} = x(6-y)$$

$$\frac{dy}{6-y} = xdx$$

$$-\int\left(\frac{1}{6-y}\right)dy = \int xdx$$

$$-\ln|6-y| = \frac{x^2}{2} + C$$

$$y - 6 = e^{-x^2/2 + C}$$

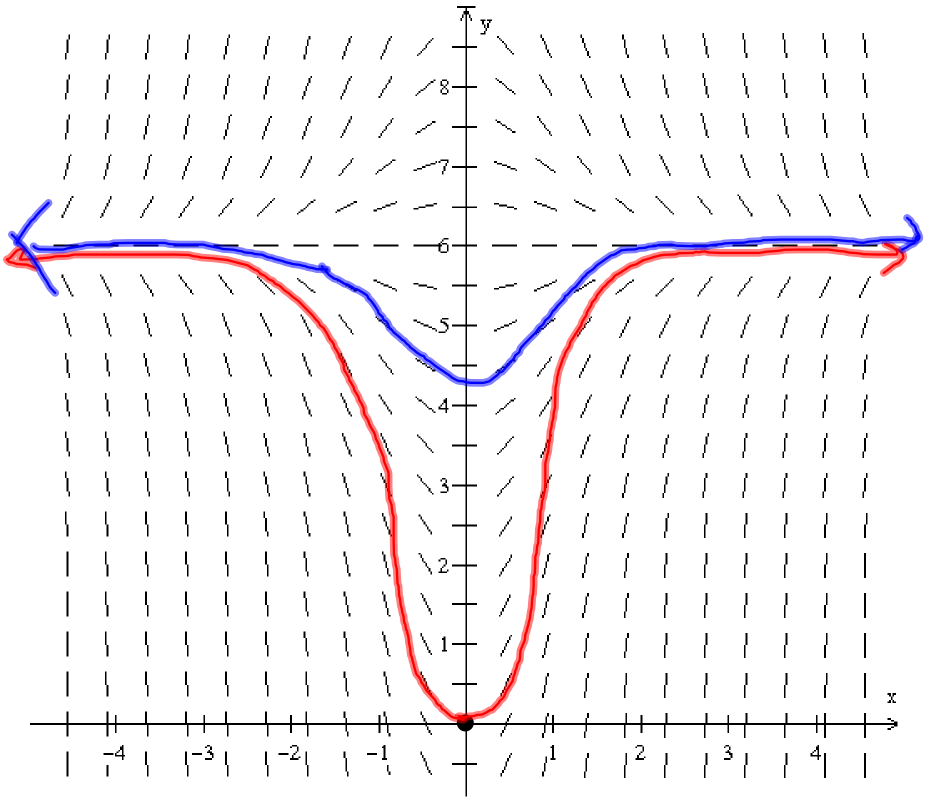
$$y = 6 + Ce^{-x^2/2}$$

$$(0,0): 0 = 6 + Ce^{-0^2/2}$$

$$0 = 6 + C$$

$$-6 = C$$

$$y = 6 - 6e^{-x^2/2}$$


y
x
8
7
6
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-2
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4

#17. Find the function y = f(t) passing through the point (0, 10) with the given first derivative. Use a graphing utility to graph the solution

$$\frac{dy}{dt} = \frac{1}{2}t$$

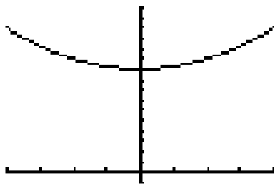
$$\int dy = \int \frac{1}{2} t\,dt$$

$$y = \frac{1}{4}t^2 + C$$

$$(0,10): 10 = \frac{1}{4}(0)^2 + C$$

$$10 = C$$

$$y = \frac{1}{4}t^2 + 10$$



Growth and Decay Models

In many applications, the rate of change of a variable y is proportional to the value of y. If y is a function of time t, the proportion can be written as follows.

Rate of change of y — is — proportional to y

$$\frac{dy}{dx} = ky$$

The general solution of this differential equation is given in the following theorem

Theorem 5.16: Exponential Growth and Decay Model

If y is a differentiable function of t such that y > 0 and y' = ky for some constant k, then

$$y = Ce^{kt}$$

C is the initial value of y, and k is the proportionality constant. Exponential growth occurs when k > 0, and exponential decay occurs when k < 0.

Example 2: Using an Exponential Growth Model

The rate of change of y is proportional to y. When t = 0, y = 2. when t = 2, y = 4. What is the value of y when t = 3?

Because y' = ky, you know that y and t are related by the equation $y = Ce^{kt}$
You can find the values of the constants C and k by applying the initial conditions

t = 0, y = 2: $2 = Ce^{0}$

$2 = C$

t = 2, y = 4: $4 = 2e^{2k}$

$2 = e^{2k}$

$\ln 2 = 2k$

$\frac{\ln 2}{2} = k \approx 0.3466$

∴ the model is $y = 2e^{0.3466t}$

when t = 3: $y = 2e^{0.3466(3)} \approx 5.657$

Radioactive decay is measured in terms of half-life - the number of years required for half o the atoms in a sample of radioactive material to decay. The half-lives of some common radioactive isotopes are as follows:

Uranium (^{238}U)	4,510,000,000 years
Plutonium (^{239}Pu)	24,360 years
Carbon (^{14}C)	5,730 years
Radium (^{226}Ra)	1,620 years
Einsteinium (^{254}Es)	270 days
Nobelium (^{257}No)	23 seconds

Example 3: Radioactive Decay

Suppose that 10 grams of the plutonium isotope Pu-239 was released in the Chernobyl nuclear accident. How long will it take for the 10 grams to decay to 1 gram?

Let y represent the mass (in grams) of the plutonium. Because the rate of decay is proportional to y, you know that $y = Ce^{kt}$

where t is the time in years. To find the values of the constants C and k, apply the initial conditions. Using the fact that y = 10 when t = 0, you can write

$$10 = Ce^{0}$$
$$10 = C$$

Now use the fact that y = 5 when t = 24,360 to find k

$$5 = 10e^{k(24,360)}$$
$$\frac{1}{2} = e^{k(24,360)}$$
$$\ln\frac{1}{2} = 24,360k$$
$$\frac{\ln\frac{1}{2}}{24,360} = k \approx -2.8454\times10^{-5}$$

$\therefore$ The model is $y = 10e^{-0.000028454t}$

When will 10 grams become 1 gram?

$$1 = 10e^{-0.000028454t}$$
$$\frac{1}{10} = e^{-0.000028454t}$$
$$\ln\frac{1}{10} = -0.000028454t$$
$$80,923\ years \approx t$$

**Note: This equation could also be written as

$$y =$$

This model is easier to derive but sometimes isn't as convenient to use

Example 4: Population Growth

Suppose an experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

Let $y = Ce^{kt}$ be the number of flies at time t, where t is measured in days. Because y = 100 when t = 2 and y = 300 when t = 4, you can write

$$100 = Ce^{2k}$$
$$C = 100e^{-2k}$$

$$300 = Ce^{4k}$$
$$300 = \left(100e^{-2k}\right)e^{4k}$$
$$300 = 100e^{2k}$$
$$3 = e^{2k}$$
$$\ln 3 = 2k$$
$$\frac{\ln 3}{2} = k \approx 0.5493$$

$$y = Ce^{0.5493t}$$
$$y = 100, t = 2 : 100 = Ce^{0.5493(2)}$$
$$\frac{\;}{e^{1.0986}} = C$$
$$33 \approx C$$

The original population (when t = 0) is approximately 33 flies

Example 5: Declining Sales

Four months after it stops advertising, a manufacturing company notices that its sales have dropped from 100,000 units per month to 80,000 units per month. If the sales follow an exponential pattern of decline, what will they be after another 2 months?

Use the exponential decay model, where t is measured in months. From the initial condition (t = 0), you know that C = 100,000. Moreover, because y = 80,000 when t = 4

$$80,000 = 100,000e^{4k}$$
$$0.8 = e^{4k}$$
$$\ln 0.8 = 4k$$
$$\frac{\ln 0.8}{4} = k \approx -0.0558$$
$$y = 100,000e^{(-0.0558)t}$$
$$t = 6 : y = 100,000e^{-0.0558(6)} \approx 71,500\, units$$

Newton's Law of Cooling: the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium.

Example 6: Newton's Law of Cooling

Let y represent the temperature (in degrees F) of an object in a room whose temperature is kept at a constant 60 degrees. If the object cools from 100 degrees to 90 degrees in 10 minutes, how much longer will it take for its temperature to decrease to 80 degrees?

From Newton's Law of Cooling, you know that the rate of change in y is proportional to the difference between y and 60. This can be written as

$$y' = k(y-60) \qquad 80 \le y \le 100$$

Now we need to solve this differential equation:

$$\frac{dy}{dt} = k(y-60)$$

$$\left(\frac{1}{y-60}\right)dy = kdt$$

$$\int \frac{1}{y-60}\,dy = \int kdt$$

$$\ln|y-60| = kt + C$$

Since y > 0, you can omit the absolute value signs

$$\ln(y-60) = kt + C$$

$$y - 60 = e^{kt+C}$$

$$y = Ce^{kt} + 60 \qquad C = eC$$

y = 100 when t = 0: $100 = Ce^{0} + 60$

$$40 = C$$

$$y = 40e^{kt} + 60$$

y = 90 when t = 10: $90 = 40e^{k(10)} + 60$

$$30 = 40e^{k(10)}$$

$$\frac{3}{4} = e^{k(10)}$$

$$\ln\frac{3}{4} = 10k$$

$$\frac{\ln\frac{3}{4}}{10} = k \approx -0.02877$$

$$y = 40e^{-0.02877t} + 60$$

$y = 80:$ $80 = 40e^{-0.02877t} + 60$

$$20 = 40e^{-0.02877t}$$

$$\frac{1}{2} = e^{-0.02877t}$$

$$\ln\frac{1}{2} = -0.02877t$$

$$t \approx 24.09\ MINUTES$$

It will require about 14.09 more minutes for the object to cool to a temperature of 80 degrees

HW p366 #4, 6, 10, 18, 22, 34, 46,